

Non-decreasable extremal Beltrami differentials of non-landslide type

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Abstract

In this paper, we deform a uniquely-extremal Beltrami differential into different non-decreasable Beltrami differentials, and then construct non-unique extremal Beltrami differentials such that they are both non-landslide and non-decreasable.

1. Introduction

Let S be a plane domain with at least two boundary points. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable domain $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are said to be equivalent, denoted by $f \sim g$, if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the boundary of S . Denote by $[f]$ the Teichmüller equivalence class of f ; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami differential of f .

Denote by $Bel(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞ -norm and by $M(S)$ the open unit ball in $Bel(S)$.

For $\mu \in M(S)$, define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

We say that μ is extremal in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$ (the corresponding quasiconformal map f is said to be extremal for its boundary values as well), uniquely extremal if $\|\nu\|_\infty > k_0(\mu)$ for any other $\nu \in [\mu]$.

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The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials on S with L^1 -norm

$$\|\varphi\| = \iint_S |\varphi(z)| dx dy < \infty.$$

In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$.

As is well known, μ is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence $\{\psi_n\} \subset Q^1(S)$, such that

$$(1.1) \quad \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S \mu \psi_n(z) dx dy = \|\mu\|_\infty.$$

By definition, a sequence $\{\psi_n\}$ is called degenerating if it converges to 0 uniformly on compact subsets of S .

We would not like to give the exact definition of Strebel point and non-Strebel point in $T(S)$. But it should be kept in mind that an extremal represents a non-Strebel point if and only if it has a degenerating Hamilton sequence (for example, see [2, 5]). We call an extremal representing a non-Strebel point to be a non-Strebel extremal.

Definition 1. *An extremal Beltrami differential μ in $Bel(S)$ is said to be of landslide type if there exists a non-empty open subset $E \subset S$ such that*

$$\operatorname{esssup}_{z \in E} |\mu(z)| < \|\mu\|_\infty;$$

otherwise, μ is said to be of non-landslide type.

The conception of *non-landslide* was firstly introduced by Li in [8]. It was proved by Fan [3] and the author [22] independently that if μ contains more than one extremal, then it contains infinitely many extremals of non-landslide type.

The following notion of *locally extremal* was introduced in [18] by Sheretov.

Definition 2. *A Beltrami differential μ in $M(S)$ is called to be locally extremal if for any domain $G \subset S$ it is extremal in its class in $T(G)$; in other words,*

$$\|\mu\|_G := \operatorname{esssup}_{z \in G} |\mu| = \sup \left\{ \frac{\operatorname{Re} \iint_G \mu \phi(z) dx dy}{\iint_G |\phi(z)| dx dy} : \phi \in Q^1(G) \right\}.$$

Obviously, the extremality for μ in S is a prerequisite condition for μ to be locally extremal. However, up to present, it is not clear whether a Teichmüller class always contains a local extremal.

Definition 3. *A Beltrami differential μ (not necessarily extremal) is called to be non-decreasable in its class $[\mu]$ if for $\nu \in [\mu]$,*

$$(1.2) \quad |\nu(z)| \leq |\mu(z)| \text{ a.e. in } S,$$

implies that $\mu = \nu$; otherwise, μ is called to be decreasable.

The notion of non-decreasable dilatation was firstly introduced by Reich in [12] when he studied the unique extremality of quasiconformal mappings. The author [20] proved that the non-decreasable extremal in a class may be non-unique. Shen and Chen [17] proved that there are infinitely many non-decreasable representatives (generally, not extremal) in a class while the existence of a non-decreasable extremal is generally unknown. It should be noted that a non-unique extremal is certainly of non-constant modulus if it is non-decreasable.

In particular, a unique extremal is naturally non-landslide, locally extremal and non-decreasable. However, it is not clear what about the converse. The following problem is posed in [22].

Problem \mathcal{A} . Is there a non-unique extremal μ such that μ is non-landslide, locally extremal and non-decreasable?

Up to present, the problem seems open. We can find some examples related to the problem in literatures. The first (even essentially only) example of non-unique extremal which is both non-landslide and locally extremal was given by Reich in [10], but the extremal is decreasable for it has a constant modulus. To get a non-unique extremal of non-constant modulus that is both non-landslide and locally extremal, one may apply the Construction Theorem in [21] in a refined manner. The second example was given by the author in Theorem 1 (2) of [20] that provides a non-unique extremal which is both locally extremal and non-decreasable but landslide.

One might expect the third example for a non-unique extremal which is both non-landslide and non-decreasable. However, no such an extremal can be found in literatures. The motivation of this paper is to construct such an example.

This paper is organized as follows. In Section 2, we introduce the Main Inequality and give its application. The Infinitesimal Main Inequality is introduced in Section 3. We construct extremals which are both non-landslide and non-decreasable in Section 4. The Construction Theorem for the desired extremals is obtained in the last Section 5.

2. Main Inequality and its application

For brevity, we restrict our consideration to $S = \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. The universal Teichmüller space $T(\mathbb{D})$ can be viewed as the set of the equivalence classes $[f]$ of quasiconformal mappings f from \mathbb{D} onto itself. So, for any quasiconformal mapping from \mathbb{D} onto itself, there is no difference between $T(\mathbb{D})$ and $T(f(\mathbb{D}))$.

The Reich-Strebel inequality, so-called Main Inequality (see [4, 15, 16]), plays an important role in the study of Teichmüller theory. To introduce the inequality, we need some denotations. Suppose that f and g are two quasiconformal mappings of \mathbb{D} onto itself with the Beltrami differentials μ, ν respectively. Let $F = f^{-1}$, $G = g^{-1}$ and $\tilde{\mu}, \tilde{\nu}$ denote the Beltrami differentials of F, G respectively. Put $\alpha = \tilde{\mu} \circ f$, $\beta = \tilde{\nu} \circ f$. Then we have

Main Inequality. If $\mu \sim \nu$, i.e., f and g are equivalent, then for any $\varphi \in Q(\mathbb{D})$,

$$(2.1) \quad \iint_{\mathbb{D}} \varphi \, dxdy \leq \iint_{\mathbb{D}} |\varphi(z)| \frac{\left| 1 - \mu(z) \frac{\varphi(z)}{|\varphi(z)|} \right|^2}{1 - |\mu(z)|^2} \frac{\left| 1 + \beta \frac{\mu}{\alpha} \frac{1 - \bar{\mu} \frac{\bar{\varphi}}{|\varphi|}}{1 - \mu \frac{\varphi}{|\varphi|}} \right|^2}{1 - |\beta|^2} \, dxdy,$$

or equivalently (see [10, 12]),

$$(2.2) \quad \operatorname{Re} \iint_{\mathbb{D}} \frac{(\beta - \alpha)(1 - \alpha\bar{\beta})\tau}{(1 - |\alpha|^2)(1 - |\beta|^2)} \varphi \, dxdy \leq \iint_{\mathbb{D}} \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi| \, dxdy,$$

where $\tau = \frac{\partial_z f}{\partial_z \bar{f}} = -\frac{\mu}{\alpha}$.

Let

$$\operatorname{sgn} z = \begin{cases} \frac{z}{|z|}, & z \neq 0, \\ 0, & z = 0 \end{cases}$$

be the signal function of $z \in \mathbb{C}$.

Lemma 2.1. *With the same notations as above, if $\mu \sim \nu$ and $|\tilde{\nu}(w)| \leq |\tilde{\mu}(w)|$ for almost every $w \in \mathbb{D}$, then there is a constant C depending only on $k = \|\mu\|_\infty$, such that for any $\varphi \in Q(\mathbb{D})$,*

$$(2.3) \quad \iint_{\Lambda} |\alpha - \beta|^2 |\varphi| \, dxdy \leq C \iint_{\Lambda} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)] \, dxdy,$$

where $\Lambda = \{z \in \mathbb{D} : \mu(z) \neq 0\}$.

Proof. Since $\alpha(z) = \tilde{\mu}(f(z)) = -\frac{\mu(z)}{\tau}$, we have $\alpha(z) = 0$ when $z \in \mathbb{D} \setminus \Lambda$. By the condition $|\tilde{\nu}(w)| \leq |\tilde{\mu}(w)|$ ($w = f(z) \in \mathbb{D}$), it forces $\beta(z) = 0$ when $z \in \mathbb{D} \setminus \Lambda$. It follows directly from (2.2) that

$$(2.4) \quad \begin{aligned} & - \iint_{\Lambda} \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi| \, dxdy \leq \operatorname{Re} \iint_{\Lambda} \frac{(\beta - \alpha)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{\mu}{\alpha} \varphi \, dxdy \\ & = -\operatorname{Re} \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} \varphi \operatorname{sgn} \mu \, dxdy. \end{aligned}$$

In order to group $|\varphi| - \varphi \operatorname{sgn} \mu$ together, we add

$$\operatorname{Re} \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} |\varphi| \, dxdy$$

to both sides of (2.4) and get

$$(2.5) \quad \begin{aligned} & \operatorname{Re} \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} |\varphi| \, dxdy - \iint_{\Lambda} \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\varphi| \, dxdy \\ & \leq \operatorname{Re} \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} (|\varphi| - \varphi \operatorname{sgn} \mu) \, dxdy. \end{aligned}$$

By a deformation, we have

$$(2.6) \quad \begin{aligned} & \iint_{\Lambda} \frac{(1 - |\alpha|)|\alpha - \beta|^2 + (1 + |\alpha|)(|\alpha|^2 - |\beta|^2)}{2|\alpha|(1 + |\alpha|)(1 - |\beta|^2)} |\varphi| \, dxdy \\ & \leq Re \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} (|\varphi| - \varphi sgn\mu) \, dxdy. \end{aligned}$$

Then,

$$(2.7) \quad \begin{aligned} & \iint_{\Lambda} \frac{(1 - |\alpha|)|\alpha - \beta|^2}{2|\alpha|(1 + |\alpha|)(1 - |\beta|^2)} |\varphi| \, dxdy \\ & \leq Re \iint_{\Lambda} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{|\alpha|}{\alpha} (|\varphi| - \varphi sgn\mu) \, dxdy. \end{aligned}$$

Since $|\beta(z)| \leq |\alpha(z)| \leq k$, one finds that a lower bound on the coefficient of $|\varphi|$ on the left of (2.6) is

$$\frac{(1 - |\alpha|)|\alpha - \beta|^2}{2|\alpha|(1 + |\alpha|)(1 - |\beta|^2)} \geq \frac{1 - k}{2k(1 + k)} |\alpha - \beta|^2.$$

An upper bound for the integrand on the right of (2.6) is

$$|\alpha - \beta| \frac{1 + |\alpha|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} ||\varphi| - \varphi sgn\mu| \leq \frac{1 + k^2}{(1 - k^2)^2} |\alpha - \beta| \cdot ||\varphi| - \varphi sgn\mu|.$$

Therefore, by the identity

$$(2.8) \quad ||w| - w|^2 = 2|w|(|w| - Rew),$$

we have

$$(2.9) \quad \begin{aligned} \iint_{\Lambda} |\alpha - \beta|^2 |\varphi| \, dxdy & \leq \frac{2k(1 + k^2)}{(1 + k)(1 - k)^3} \iint_{\Lambda} |\alpha - \beta| \cdot ||\varphi| - \varphi sgn\mu| \, dxdy \\ & = C' \iint_{\Lambda} |\alpha - \beta| |\varphi|^{\frac{1}{2}} [|\varphi| - Re(\varphi sgn\mu)]^{\frac{1}{2}} \, dxdy, \end{aligned}$$

where $C' = C'(k) = \frac{2\sqrt{2}k(1+k^2)}{(1+k)(1-k)^3}$.

Applying Schwarz's Inequality, we get

$$\iint_{\Lambda} |\alpha - \beta|^2 |\varphi| \, dxdy \leq C'^2 \iint_{\Lambda} [|\varphi| - Re(\varphi sgn\mu)] \, dxdy.$$

□

3. Infinitesimal Main Inequality

Two Beltrami differentials μ and ν in $Bel(S)$ are said to be infinitesimally equivalent, denoted by $\mu \approx \nu$, if

$$\int_S \mu \varphi = \int_S \nu \varphi, \text{ for any } \varphi \in Q(S).$$

The tangent space $B(S)$ of $T(S)$ at the basepoint is defined as the set of the quotient space of $Bel(S)$ under the equivalence relation. Denote by $[\mu]_B$ the equivalence class of μ in $B(S)$. The set of all Beltrami differentials equivalent to zero is called the \mathcal{N} -class in $Bel(S)$.

We say that μ is (infinitesimally) extremal (in $[\mu]_B$) if $\|\mu\|_\infty = \|[\mu]_B\|$, (infinitesimally) uniquely extremal if $\|\nu\|_\infty > \|\mu\|_\infty$ for any other $\nu \in [\mu]_B$.

The notion of infinitesimal Strebel point and non-Strebel point can be found in [2]. Any extremal in an infinitesimal non-Strebel point is called an infinitesimal non-Strebel extremal.

Definition 4. A Beltrami differential μ (not necessarily extremal) is called to be infinitesimally non-decreasable in its class $[\mu]_B$ if for $\nu \in [\mu]_B$,

$$(3.1) \quad |\nu(z)| \leq |\mu(z)| \text{ a.e. in } S,$$

implies that $\mu = \nu$; otherwise, μ is called to be infinitesimally decreasable.

The following is the Infinitesimal Main Inequality on \mathbb{D} , whose proof can be found in [1, 10].

Infinitesimal Main Inequality. Suppose $\mu, \nu \in M(\mathbb{D})$. If $\mu \approx \nu$, i.e., μ and ν are infinitesimally equivalent, then for any $\varphi \in Q(\mathbb{D})$,

$$(3.2) \quad \operatorname{Re} \iint_{\mathbb{D}} \frac{(\mu - \nu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \varphi \, dx dy \leq \iint_{\mathbb{D}} \frac{|\mu - \nu|^2 |\nu|}{1 - |\nu|^2} |\varphi| \, dx dy.$$

Lemma 3.1. Let $\mu, \nu \in B(S)$. If $\mu \approx \nu$ and $|\nu(z)| \leq |\mu(z)|$ for almost every $z \in \mathbb{D}$, then there is a constant C depending only on $k = \|\mu\|_\infty$, such that for any $\varphi \in Q(\mathbb{D})$,

$$(3.3) \quad \iint_{\Lambda} |\alpha - \beta|^2 |\varphi| \, dx dy \leq C \iint_{\Lambda} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)] \, dx dy,$$

where $\Lambda = \{z \in \mathbb{D} : \mu(z) \neq 0\}$.

Proof. At first, let $k < 1$. Since $|\nu(z)| \leq |\mu(z)| = 0$ when $z \in \mathbb{D} \setminus \Lambda$, it follows from (3.2) that

$$(3.4) \quad - \iint_{\Lambda} \frac{|\mu - \nu|^2 |\nu|}{1 - |\nu|^2} |\varphi| \, dx dy \leq \operatorname{Re} \iint_{\Lambda} \frac{(\nu - \mu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \varphi \, dx dy.$$

Hence,

$$(3.5) \quad - \iint_{\Lambda} \frac{|\mu - \nu|^2 |\mu|}{1 - |\nu|^2} |\varphi| \, dx dy \leq \operatorname{Re} \iint_{\Lambda} \frac{(\nu - \mu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \varphi \, dx dy.$$

In order to group $|\varphi| - \varphi \operatorname{sgn} \mu$ together, we add

$$\operatorname{Re} \iint_{\mathbb{D}} \frac{(\mu - \nu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \frac{|\mu|}{\mu} |\varphi| \, dx dy$$

to both sides of (3.5) and get

$$(3.6) \quad \begin{aligned} & \operatorname{Re} \iint_{\mathbb{D}} \frac{(\mu - \nu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \frac{|\mu|}{\mu} |\varphi| \, dxdy - \iint_{\Lambda} \frac{|\mu - \nu|^2 |\nu|}{1 - |\nu|^2} |\varphi| \, dxdy \\ &= \operatorname{Re} \iint_{\Lambda} \frac{(\nu - \mu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \frac{|\mu|}{\mu} (|\varphi| - \varphi \operatorname{sgn} \mu) \, dxdy. \end{aligned}$$

By a deformation, we have

$$(3.7) \quad \begin{aligned} & \iint_{\Lambda} \frac{(1 - |\mu|^2)|\mu - \nu|^2 + (1 - |\mu|^2)(|\mu|^2 - |\nu|^2)}{2|\mu|(1 - |\nu|^2)} |\varphi| \, dxdy \\ & \leq \operatorname{Re} \iint_{\Lambda} \frac{(\mu - \nu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \frac{|\mu|}{\mu} (|\varphi| - \varphi \operatorname{sgn} \mu) \, dxdy. \end{aligned}$$

Then,

$$(3.8) \quad \begin{aligned} & \iint_{\Lambda} \frac{(1 - |\mu|^2)|\mu - \nu|^2}{2|\mu|(1 - |\nu|^2)} |\varphi| \, dxdy \\ & \leq \operatorname{Re} \iint_{\Lambda} \frac{(\mu - \nu)(1 - \mu\bar{\nu})}{1 - |\nu|^2} \frac{|\mu|}{\mu} (|\varphi| - \varphi \operatorname{sgn} \mu) \, dxdy. \end{aligned}$$

Since $|\nu(z)| \leq |\mu(z)|$, one finds that a lower bound on the coefficient of $|\mu - \nu|^2 |\varphi|$ on the left of (3.8) is

$$\frac{1 - |\mu|^2}{2|\mu|(1 - |\nu|^2)} \geq \frac{1 - k^2}{2k}.$$

An upper bound of the integrand on the right side of (3.8) is

$$|\mu - \nu| \frac{1 + |\mu|^2}{1 - |\mu|^2} ||\varphi| - \varphi \operatorname{sgn} \mu| \leq \frac{1 + k^2}{1 - k^2} |\mu - \nu| \cdot ||\varphi| - \varphi \operatorname{sgn} \mu|.$$

Therefore, using the identity (2.8), we get

$$(3.9) \quad \begin{aligned} & \iint_{\Lambda} |\mu - \nu|^2 |\varphi| \, dxdy \leq \frac{2k(1 + k^2)}{(1 - k^2)^2} \iint_{\Lambda} |\mu - \nu| |\varphi| - \varphi \operatorname{sgn} \mu| \, dxdy \\ &= \tilde{C} \iint_{\Lambda} |\mu - \nu| |\varphi|^{\frac{1}{2}} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)]^{\frac{1}{2}} \, dxdy, \end{aligned}$$

where $\tilde{C} = \tilde{C}(k) = \frac{2\sqrt{2}k(1+k^2)}{(1-k^2)^2}$.

Applying Schwarz's Inequality, we obtain

$$(3.10) \quad \iint_{\Lambda} |\mu - \nu|^2 |\varphi| \, dxdy \leq \tilde{C}^2 \iint_{\Lambda} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)] \, dxdy.$$

Now, if $k \geq 1$. Let $\mu_1 = \frac{\mu}{sk}$, $\nu_1 = \frac{\nu}{sk}$ where $s > 1$. Then $\mu_1 \approx \nu_1$ and $|\nu_1| \leq |\mu_1| \leq \frac{1}{s}$ for almost all $z \in \mathbb{D}$. It derives from (3.10) that

$$(3.11) \quad \iint_{\Lambda} |\mu - \nu|^2 |\varphi| \, dxdy \leq \frac{8k^2 s^4 (s^2 + 1)^2}{(s^2 - 1)^4} \iint_{\Lambda} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)] \, dxdy.$$

By a refined computation, one can show that for any $k \geq 0$,

$$(3.12) \quad \iint_{\Lambda} |\mu - \nu|^2 |\varphi| \, dxdy \leq 8k^2 \iint_{\Lambda} [|\varphi| - \operatorname{Re}(\varphi \operatorname{sgn} \mu)] \, dxdy.$$

□

4. Non-decreasable extremals of non-landslide type

In this section, we deform a non-Strebel unique extremal into an extremal in a way that keeps “non-landslide” and “non-decreasable”. We need to use the Characterization Theorem (see Theorem 1 in [1]) on the unique extremality. Before stating the theorem, we interpret what the Reich’s condition and Reich sequence are.

Following [1], we say that $\mu \in \operatorname{Bel}(\mathbb{D})$ satisfies Reich’s condition on a subset $\mathcal{S} \subset \mathbb{D}$ if there exists a sequence $\{\varphi_n\}$ in $Q(\mathbb{D})$ such that

- (a) $\delta[\varphi_n] := \|\mu\|_{\infty} \|\varphi_n\| - \operatorname{Re} \iint_{\mathbb{D}} \mu(z) \varphi_n(z) \, dxdy \rightarrow 0$, and
- (b) $\liminf_{n \rightarrow \infty} |\varphi_n(z)| > 0$ for almost all $z \in \mathcal{S}$.

Generally, if μ satisfies Reich’s condition above, we call $\{\varphi_n\}$ a Reich sequence for μ on \mathcal{S} .

The Characterization Theorem discloses the relationship among unique extremality infinitesimal, unique extremality and Reich’s condition.

Theorem A. *Let $\mu \in M(\mathbb{D})$ with a constant modulus. Then the following three conditions are equivalent:*

- (i) μ is uniquely extremal in its class in $T(\mathbb{D})$;
- (ii) μ is uniquely extremal in its class in $B(\mathbb{D})$;
- (iii) μ satisfies Reich’s condition on \mathbb{D} , i.e. μ has a Reich sequence on \mathbb{D} .

The following theorem deforms a unique extremal Beltrami differential into a non-decreasable Beltrami differential which generally does not keep the extremality.

Theorem 4.1. *Suppose $\eta \in M(\mathbb{D})$ is uniquely extremal in $[\eta]$ and has a constant modulus. Let $k = \|\eta\|_{\infty}$. Put*

$$\mu(z) = \kappa(z)\eta(z), \quad z \in \mathbb{D},$$

where $\kappa(z)$ is a non-negative measurable function on \mathbb{D} with $\|\kappa\|_{\infty} \leq k$. Let f be the quasiconformal mapping from \mathbb{D} onto itself with the Beltrami differential μ . Then $F = f^{-1}$ has a non-decreasable Beltrami differential $\tilde{\mu}$ in its Teichmüller class $[\tilde{\mu}]$.

Proof. To avoid triviality, assume $k > 0$. For any given $g \in [f]$, let $G = g^{-1}$. Let ν and $\tilde{\nu}$ denote the Beltrami differentials of g and G respectively. To prove that $\tilde{\mu}$ is non-decreasable in its Teichmüller class $[\tilde{\mu}]$, it is sufficient to show that if $|\tilde{\nu}(w)| \leq |\tilde{\mu}(w)|$ holds for almost all $w \in \mathbb{D}$, then $\tilde{\mu} = \tilde{\nu}$. Use the denotations $\alpha = \tilde{\mu} \circ f$, $\beta = \tilde{\nu} \circ f$.

On the one hand, since μ is uniquely extremal and has constant absolute value on \mathbb{D} , by Theorem A μ has a Reich sequence on \mathbb{D} , that is, there is a sequence $\{\varphi_n\} \subset Q(\mathbb{D})$ such that

- (a) $\delta[\varphi_n] = k\|\varphi_n\| - \operatorname{Re} \iint_{\mathbb{D}} \mu(z) \varphi_n(z) dx dy \rightarrow 0$, and
 (b) $\liminf_{n \rightarrow \infty} |\varphi_n(z)| > 0$ for almost all $z \in \mathbb{D}$.

On the other hand, by Lemma 2.1 and Reich's condition (a), when $|\tilde{\nu}(w)| \leq |\tilde{\mu}(w)|$ holds for almost all $w \in \mathbb{D}$, we have

$$(4.1) \quad \begin{aligned} \iint_{\Lambda} |\alpha - \beta|^2 |\varphi_n| dx dy &\leq C \iint_{\Lambda} [|\varphi_n| - \operatorname{Re}(\varphi_n \operatorname{sgn} \mu)] dx dy \\ &= \frac{C}{k} \iint_{\Lambda} [k|\varphi_n| - \operatorname{Re} \mu(z) \varphi_n(z)] dx dy \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

where $\Lambda = \{z \in \mathbb{D} : \mu(z) \neq 0\}$ and C is a constant depending only on k . It follows from Reich's condition (b) and Fatou's Lemma that $\alpha = \beta$ a.e. on Λ . Hence, $\tilde{\mu}(w) = \tilde{\nu}(w)$ for almost all $w \in \mathbb{D}$. □

The second theorem deforms a unique extremal Beltrami differential into a non-landslide and non-decreasable Beltrami differential which keeps the extremality.

Theorem 4.2. *Suppose $\eta \in M(\mathbb{D})$ is uniquely extremal such that $[\eta]$ is not a Strebel point. Assume in addition that η has a constant modulus. Let $k = \|\eta\|_{\infty} > 0$. Suppose $E \subset \mathbb{D}$ is a compact subset with positive measure and empty interior. Let κ be a non-negative measurable function on \mathbb{D} such that $\kappa(z) = 1$ for $z \in \mathbb{D} \setminus E$ and $\operatorname{esssup}_{z \in E} |\kappa| < 1$. Put*

$$(4.2) \quad \mu(z) = \kappa(z) \eta(z), \quad z \in \mathbb{D},$$

Let f be the quasiconformal mapping from \mathbb{D} onto itself with the Beltrami differential μ . Then the Beltrami differential $\tilde{\mu}$ of $F = f^{-1}$ is extremal, non-landslide and non-decreasable in its Teichmüller class $[\tilde{\mu}]$.

Proof. At first, by Theorem 4.1, $\tilde{\mu}$ is non-decreasable in $[\tilde{\mu}]$. Since η is a non-Strebel extremal, η has a degenerating Hamilton sequence $\{\phi_n\} \subset Q^1(\mathbb{D})$. Noting $\mu(z) = \eta(z)$ for $z \in \mathbb{D} \setminus E$, it is easy to see that $\{\phi_n\}$ is also a Hamilton sequence for μ and hence μ is extremal. Furthermore, $\tilde{\mu}$ is extremal. It is easy to verify that $|\tilde{\mu}(w)| = k$ for $w \in \mathbb{D} \setminus f(E)$ and $\operatorname{esssup}_{w \in f(E)} \tilde{\mu} = \operatorname{esssup}_{z \in E} |\kappa| < 1$. Because $f(E)$ has empty interior, by definition it is obvious that μ is non-landslide. □

The following two theorems are the counterparts of Theorems 4.1 and 4.2 in the infinitesimal case, respectively.

Theorem 4.3. *Suppose $\eta \in \operatorname{Bel}(\mathbb{D})$ is infinitesimally uniquely extremal in $[\eta]$ and has a constant modulus. Let $k = \|\eta\|_{\infty}$. Put*

$$\mu(z) = \kappa(z) \eta(z), \quad z \in \mathbb{D},$$

where $\kappa(z)$ is a non-negative measurable function on \mathbb{D} with $\|\kappa\|_{\infty} \leq k$. Then μ is infinitesimally non-decreasable in its infinitesimal class $[\mu]_B$.

Proof. When $k = 0$, the proof is trivial. Now assume $k > 0$. To prove that μ is non-decreasable in its infinitesimal class $[\mu]_B$, it suffices to show that, for any given $\nu \in [\mu]_B$, if $|\nu(z)| \leq |\mu(z)|$ holds for almost all $z \in \mathbb{D}$, then $\mu = \nu$.

By the Characterization Theorem (see Theorem 1 in [1]), since μ is infinitesimally uniquely extremal, it has a Reich sequence on \mathbb{D} , that is, there is a sequence $\{\varphi_n\} \subset Q(\mathbb{D})$ such that

- (a) $\delta[\varphi_n] = k\|\varphi_n\| - \operatorname{Re} \iint_{\mathbb{D}} \mu(z) \varphi_n(z) dx dy \rightarrow 0$, and
- (b) $\liminf_{n \rightarrow \infty} |\varphi_n(z)| > 0$ for almost all $z \in \mathbb{D}$.

On the other hand, by Lemma 3.1 and Reich's condition (a), when $|\nu(z)| \leq |\mu(z)|$ holds for almost all $z \in \mathbb{D}$, we have

$$(4.3) \quad \begin{aligned} \iint_{\Lambda} |\mu - \nu|^2 |\varphi_n| dx dy &\leq C \iint_{\Lambda} [|\varphi_n| - \operatorname{Re}(\varphi_n \operatorname{sgn} \mu)] dx dy \\ &= \frac{C}{k} \iint_{\Lambda} [k|\varphi_n| - \operatorname{Re} \mu(z) \varphi_n(z) dx dy] \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

where $\Lambda = \{z \in \mathbb{D} : \mu(z) \neq 0\}$ and C is a constant depending only on k . It follows from Reich's condition (b) and Fatou's Lemma that $\mu = \nu$ a.e. on Λ . Hence, $\mu(z) = \nu(z)$ for almost all $z \in \mathbb{D}$. □

Theorem 4.4. *Suppose $\eta \in \operatorname{Bel}(\mathbb{D})$ is uniquely extremal such that $[\eta]_B$ is a not an infinitesimal Strebel point. Assume in addition that η has a constant modulus. Let $k = \|\eta\|_{\infty} > 0$. Suppose $E \subset \mathbb{D}$ is a compact subset with positive measure and empty interior. Let κ be a non-negative measurable function on \mathbb{D} such that $\kappa(z) = 1$ for $z \in \mathbb{D} \setminus E$ and $\operatorname{esssup}_{z \in E} |\kappa| < 1$. Put*

$$\mu(z) = \kappa(z) \eta(z), \quad z \in \mathbb{D},$$

Then the Beltrami differential μ is extremal, non-landslide and infinitesimally non-decreasable in its infinitesimal class $[\mu]_B$.

Proof. At first, by Theorem 4.3, μ is non-decreasable in $[\mu]_B$. Since η is an infinitesimal non-Strebel extremal and $\mu(z) = \eta(z)$ for $z \in \mathbb{D} \setminus E$, it is easy to see that μ is extremal. Notice that $|\mu(z)| = k$ for $z \in \mathbb{D} \setminus E$ and $\operatorname{esssup}_{z \in E} |\kappa| < 1$. Because E has empty interior, by definition it is obvious that μ is infinitesimally non-landslide. □

5. Construction Theorem

Using Theorem 4.2, we can get extremal Beltrami differential $\tilde{\mu}$ that is both non-landslide and non-decreasable. But it is not sure whether $\tilde{\mu}$ is not uniquely extremal. To ensure that $\tilde{\mu}$ is a non-unique extremal in addition, we need to choose E in Theorem 4.2 carefully. A 2-dimensional Cantor set \mathcal{C} in \mathbb{D} with non-zero measure is constructed for the requirement in general case.

We construct a so-called $\frac{1}{5}$ -Cantor set in the closed, bounded interval $I = [0, 1]$ at first. The first step in the construction is to subdivide I into five intervals of equal

length $\frac{1}{5}$ and remove the interior of the middle interval, that is, we remove the interval $(\frac{2}{5}, \frac{3}{5})$ from the interval $[0, 1]$ to obtain the closed set C_1 , which is the union of two disjoint closed intervals, each of length $\frac{2}{5}$:

$$C_1 = [0, \frac{2}{5}] \cup [\frac{3}{5}, 1].$$

We now repeat this “open middle $\frac{1}{5}$ – removal” on each of the two intervals in C_1 to obtain a closed set C_2 , which is the union of 2^2 closed intervals, each of length $\frac{2^2}{5^2}$:

$$C_2 = [0, \frac{4}{5^2}] \cup [\frac{6}{5^2}, \frac{2}{5}] \cup [\frac{3}{5}, \frac{19}{5^2}] \cup [\frac{21}{5^2}, 1].$$

We now repeat this “open middle $\frac{1}{5}$ – removal” on each of the two intervals in C_2 to obtain a closed set C_3 , which is the union of 2^3 closed intervals, each of length $\frac{2^3}{5^3}$. We continue the removal operation countably many times to obtain the countable collection of sets $\{C_k\}_{k=1}^\infty$. We define the $\frac{1}{5}$ -Cantor set \mathcal{C} by

$$\mathcal{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^\infty$ possesses the following properties:

- (i) $\{C_k\}_{k=1}^\infty$ is a descending sequence of closed sets;
- (ii) For each k , C_k is the disjoint union of 2^k closed intervals, each of length $\frac{2^k}{5^k}$.

It is easy to compute the measure of \mathcal{C} :

$$meas(\mathcal{C}) = 1 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{5^k} = \frac{2}{3}.$$

Given $\lambda \in (0, 1)$, let $\mathcal{C}_\lambda = \lambda\mathcal{C} =: \{\lambda x, x \in \mathcal{C}\}$ and $\mathcal{C} = \{re^{i\theta} : r \in \mathcal{C}_\lambda, \theta \in [0, 2\pi)\}$. Then \mathcal{C} is 2-dimensional Cantor set in \mathbb{D} with empty interior and $meas(\mathcal{C}) > 0$.

Construction Theorem I. (1) Replace E by \mathcal{C} and keep other assumptions in Theorem 4.2. Then $\tilde{\mu}$ is a non-unique extremal that is both non-landslide and non-decreasable.

(2) Replace E by \mathcal{C} and keep other assumptions in Theorem 4.4. Then μ is a non-unique extremal that is both non-landslide and infinitesimally non-decreasable.

Proof. By the analysis above, we only need to show that μ is not uniquely extremal in both cases. In virtue of Theorem A, it is sufficient and more convenient to prove that μ is not infinitesimally uniquely extremal.

Recall that \mathcal{N} is the collection of Beltrami differentials infinitesimally equivalent to 0. Let $\zeta \in Bel(\mathbb{D})$ and define the support set of ζ by $supp(\zeta) := \{z \in \mathbb{D} : \zeta(z) \neq 0\}$. Set

$$Z[\mathcal{C}] := \{\zeta \in \mathcal{N} : supp(\zeta) \subset \mathcal{C}\}.$$

It is obvious that $0 \in Z[\mathcal{C}]$. If $Z[\mathcal{C}] \setminus \{0\} \neq \emptyset$, then for any $\gamma \in Z[\mathcal{C}] \setminus \{0\}$, $\mu + t\gamma \in [\mu]_B$ for any $t \in \mathbb{C}$. Observe the condition $\text{esssup}_{z \in \mathcal{C}} |\mu| < k$. Then, $\mu + t\gamma$ is extremal in

$[\mu]_B$ when $|t|$ is sufficiently small which implies that μ is a non-unique extremal. It remains to show that $Z[\mathcal{C}] \setminus \{0\} \neq \emptyset$. Fix a positive integer number m and let

$$\gamma(z) = \begin{cases} z^m, & z \in \mathcal{C}, \\ 0, & z \in \mathbb{D} \setminus \mathcal{C}. \end{cases}$$

Claim. $\gamma \in Z[\mathcal{C}] \setminus \{0\}$.

By the definition of \mathcal{N} , we need to show that

$$\iint_{\mathbb{D}} \gamma(z) \varphi(z) \, dxdy = 0, \text{ for any } \varphi \in Q(\mathbb{D}).$$

Note that $\{1, z, z^2, \dots, z^n, \dots\}$ is a base of the Banach space $Q(\mathbb{D})$. It suffices to prove

$$(5.1) \quad \iint_{\mathbb{D}} \gamma(z) z^n \, dxdy = 0, \text{ for any } n \in \mathbb{N}.$$

By the construction of \mathcal{C} , we see that the open set $A = [0, 1] \setminus \mathcal{C}$ is the union of countably many disjoint open intervals. Let $\mathcal{A} = \lambda A := \{\lambda z : z \in A\}$. Then $[0, \lambda] = \mathcal{A} \cup \mathcal{C}_\lambda$. Set $\mathcal{D} = \{re^{i\theta} : r \in \mathcal{A}, \theta \in [0, 2\pi)\}$. It is clear that

$$\mathcal{D} \cup \mathcal{C} = \{re^{i\theta} : r \in [0, \lambda], \theta \in [0, 2\pi)\} = \{z : |z| \leq \lambda\}.$$

Define

$$\tilde{\gamma}(z) := \begin{cases} z^m, & z \in \mathcal{D}, \\ 0, & z \in \mathbb{D} \setminus \mathcal{D}, \end{cases}$$

and

$$(5.2) \quad \Gamma(z) := \gamma(z) + \tilde{\gamma}(z) = \begin{cases} z^m, & |z| \leq \lambda, \\ 0, & \lambda < |z| < 1. \end{cases}$$

A simple computation shows that

$$(5.3) \quad \begin{aligned} \iint_{\mathbb{D}} \Gamma(z) z^n \, dxdy &= \iint_{|z| \leq \lambda} \Gamma(z) z^n \, dxdy = \iint_{|z| \leq \lambda} z^{m+n} \, dxdy \\ &= \int_0^\lambda r \, dr \int_0^{2\pi} e^{i(m+n)\theta} \, d\theta = 0, \text{ for any } n \in \mathbb{N}. \end{aligned}$$

Observe that \mathcal{D} is the union of countably many disjoint ring domains each of which can be written in the form $R = \{re^{i\theta} : r \in (x, x'), \theta \in [0, 2\pi)\}$, $x, x' \in (0, \lambda)$. A similar computation gives

$$(5.4) \quad \iint_R \Gamma(z) z^n \, dxdy = \iint_R z^{m+n} \, dxdy = 0, \text{ for any } n \in \mathbb{N}.$$

Hence, we get

$$(5.5) \quad \iint_{\mathcal{D}} \Gamma(z) z^n \, dxdy = 0, \text{ for any } n \in \mathbb{N}.$$

Combining (5.2), (5.3) and (5.5), we obtain

$$(5.6) \quad \iint_{\mathcal{C}} \Gamma(z) z^n dx dy = 0, \quad \text{for any } n \in \mathbb{N},$$

which is equivalent to (5.1). This completes the proof of Construction Theorem. \square

Let φ be a holomorphic function on \mathbb{D} and $\eta = k \frac{\bar{\varphi}}{|\varphi|}$. In one case, by the result in [7], for φ in a dense subset of $Q(\mathbb{D})$, the corresponding Teichmüller differential η is a non-Strebel extremal (necessarily uniquely extremal). In other case, there are a lot of holomorphic functions in \mathbb{D} with $\iint_{\mathbb{D}} |\varphi| dx dy = \infty$ such that μ is uniquely extremal, of course a non-Strebel extremal (see [6, 11, 19]), for example, let $\varphi = \frac{1}{(1-z)^2}$.

Lemma 5.1. *Let E be a compact subset of \mathbb{D} , $G = \mathbb{D} \setminus E$ and φ a holomorphic function on G . Suppose that*

(a) *μ is uniquely extremal on \mathbb{D} ,*

(b) *$\mu = \kappa(z) \frac{\overline{\varphi(z)}}{|\varphi(z)|}$ on G ,*

where κ is non-negative measurable function on G . Then

(i) *φ has a holomorphic extension $\tilde{\varphi}$ from G to \mathbb{D} ,*

(ii) *$\mu = k |\tilde{\varphi}| / \tilde{\varphi}$ a.e. in \mathbb{D} ($k = \|\mu\|_{\infty}$).*

Proof. It is a simple corollary of Theorem G4 (the Second Removable Singularity Theorem) of [9] or Theorem 2.3 on page 113 in [14]. \square

In the following Construction Theorem II, we only assume that E is a compact subset of \mathbb{D} with positive measure, provided that η is a uniquely extremal Teichmüller differential representing a non-Strebel point.

Construction Theorem II. Assume that φ is a holomorphic function on \mathbb{D} such that $\eta = k \frac{\bar{\varphi}}{|\varphi|}$ is uniquely extremal and represents a non-Strebel point.

(1) Keep other assumptions in Theorem 4.2. Then $\tilde{\mu}$ is a non-unique extremal that is both non-landslide and non-decreasable.

(2) Keep other assumptions in Theorem 4.4. Then μ is a non-unique extremal that is both non-landslide and infinitesimally non-decreasable.

Proof. It is sufficient to prove that μ is not uniquely extremal on \mathbb{D} . Actually, if μ is uniquely extremal, then by Lemma 5.1, $\mu = k \frac{\bar{\varphi}}{|\varphi|} = \eta$ on \mathbb{D} , which contradicts the assumption. \square

The non-unique extremal μ given by Construction Theorem II is not locally extremal since otherwise by Theorem G3 (the First Removable Singularity Theorem) of [9], μ is identical to η on \mathbb{D} . However, we do not know whether the non-unique extremal μ given by Construction Theorem I is possibly locally extremal. If yes, then Problem \mathcal{A} is solved.

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